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ON THE RULES OF SUPPOSITIONS IN FORMAL LOGIC

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§ 1. The theory of deduction based on the method of suppositions.

In 1926 Prof. J. Ł u k a s i e w i c z called attention to the fact that mathematicians in their proofs do not appeal to the theses of the theory of deduction, but make use of other methods of reasoning. The chief means employed in their method ist that of an arbitrary supposition. The problem raised by Mr. Ł u-k a s i e w i c z was to put these methods unter the form of structural rules and to analyze their relation to the theory of deduction. The present paper contains the solution of that problem.[1])

Here we consider as structural rules only those which refer to the external appearance of expressions. It is possible to formulate such rules only for a formal system in which all propositions are written in symbols. In the present paper, we shall use Mr. Ł u k a s i e w i c z's [2]) bracket - free symbolism. The implication „if α, then β" will be symbolized by „$C\alpha\beta$" and the negation „not α" by „$N\alpha$". In the above, α und β stand for significant expressions of the system. The significant expression built up by means of „C" and „N" can be defined

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[1]) The first results on that subject obtained by the author in 1926 at Prof. Ł u k a s i e w i c z's seminar were presented at the First Polish Mathematical Congres in Lwów in 1927 and were mentioned in the proceedings of the Congress: *Kalçga pamiqtkowa pierwszego polskiego zjazdu matematycznego*, Kraków 1929.

J. Ł u k a s i e w i c z, *Elementy logiki matematycznej*. Opracował M. Presburger, Warszawa 1929 (litogr.) p. 38 *ss*. - J. Ł u k a s i e w i c z und A. Tarksi. *Untersuchungen über den Aussagenkalkül*. Comptes Rendus des séances de la Soc. des Sciences et des Lettres de Varsovie, XXIII. 1930. Classe III.

by the two following conditions: (1) such an expression contains more variables than symbols "*C*", (2) no initial part of this expression contains more variables than symbols "*C*". In condition (2), we must take "part" in the sense of *proper* part, *i.e.* as distinct from the whole. As to the variables, we shall make use of small Latin lettres "*p*", "*q*", "*r*" etc. In accordance with the above explanations, the symbolic expression "*CpCCpqq*" can be read: "If *p*, then, if *p* implies *q, q*".

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We intend to analyze a practical proof by making use of the method of suppositions. How can we convince ourselves of the truth of the proposition *"CpCCpqq"?* We shall do it as follows.

Suppose p. This supposition being granted *suppose Cpq*. Thus we have assumed "*p*" and "if *p*, then *q"*. Hence *q* follows. We then observe that *"q"* is a consequence of the supposition *"Cpq",* and obtain as a deduction: "if *p* implies *q,* then *q" i.e. CCpqq*. Thus having supposed "*p*", we have deduced this last proposition; from this fact, we can infer *CpCCpqq*.

This last proposition does not depend upon any supposition. It would remain true even in case the suppositions used above should be false. All such processes as the above grow clearer, when we introduce prefixes denoting which propositions are consequences of a given supposition. These prefixes will contain numbers classifying the suppositions; thus the number "*1*" will correspond to our first supposition "*p*". Such a number must be written before the supposition to which it corresponds and before all expressions which are assumed under the conviction that this supposition is true. One of the expressions written within the scope of validity of the first supposition was "*suppose Cpq",* its prefix therefore must begin with the number "*1*". On the other hand "*Cpq*", being a supposition itself, will obtain its own number which will take the second place in the prefix. *"Cpq",* being the first supposition made in the scope of the former one, will also have "*1*" as its number, but its prefix will be different, namely "*1·1·*". The dots are adjoined to the prefix in order to remove ambiguity. The word "*suppose*" of the usual language will be symbolized by the letter "*S*" written down immediately after the prefix. The above conventions lead us to some new expressions which must be considered as significant ones. In the following explanations, however we shall retain for the term "proposition" the meaning already given, namely the significant proposition of the usual theory of deduction as above defined. Thus our sketch of demonstration takes the following form:

$1 \cdot Sp$
$1 \cdot 1 \cdot SCpq$
$1 \cdot 1 \cdot q$
$1 \cdot CCpqq$
$CpCCpqq$

The reader will easily understand the following process:

$2 \cdot SCNpNq$
$2 \cdot 1 \cdot Sq$
$2 \cdot 1 \cdot 1 \cdot SNp$
$2 \cdot 1 \cdot 1 \cdot Nq$

The supposition "*Np*" with the prefix "$2 \cdot 1 \cdot 1 \cdot$" leads us to a contradiction consisting on the simultaneous validity of "*q*" and "*Nq*". We can therefore deduce:

$2 \cdot 1 \cdot p$
$2 \cdot Cqp$
$CCNpNqCqp$[3]

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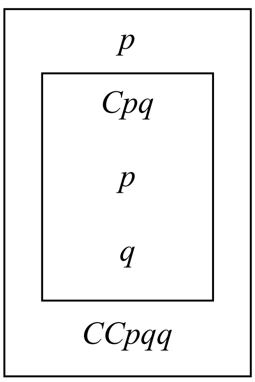
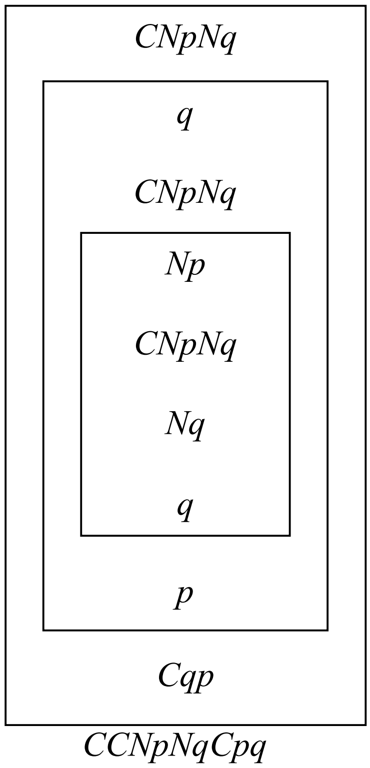
[3]) In 1926 the system was expressed in another symbolism, which is shown below for the same examples of reasoning:



CpCCpqq

We now intend to make possible the assuming of the above expressions for theses of a deductive system whose structural rules we shall formulate. In the formulation of these rules, we shall make use of some abbreviated modes of speaking about the expressions written in the system. In order to avoid any misunderstanding, we must always remember that, by an expression, a thesis etc., we shall treat a given inscription as a material object, just as Prof. S. L e ś n i e w s k i did in the explanations concerning his systems[4]). Thus two inscriptions having the same appearance but writ-



Certain expressions as "*p*" "*q*", "*CNpNq*" which have been written outside of some rectangles, have been repeated inside of them. In doing so, we obeyed a particular rule which now, through the modification of others, has become superfluous.

[4]) S. L e ś n i e w s k i. *Grundzüge eines neuen Systems der Grundlagen der Mathematik.* Fundamenta Mathematicae. Vol. XIV. 1929 p. 59 *ss*. S. L e ś n i e w s k i. *Über die Grundlagen der Ontologie.* Comptes Rendus d.s. de la Soc. des Sciences et des Lettres de Varsovie. XXIII. 1930. C1. III p. 115 *ss*. We use the terms "thesis", "system" with the same meaning as Mr. L e ś n i e w s k i does.

ten down in different places must never be taken as identical; they can only be said to be *equiform w*ith each other.

In order to be able to draw examples illustrative of the conventions we shall make, we may suppose that the steps of proofs written above are already theses of the system and that no other thesis exists. Consider a thesis containing a supposition, e.g. the thesis having the form "*2·1·Sq*". *All* other theses having their initial parts equiform with the prefix "*2·1·*" are those which have the following forms: "*2·1·1·SNp*", "*2·1·1·Nq*", and "*2:1:p*". The class composed of a supposition α and of all expressions which in other theses are preceded by initial parts equiform to the prefix of *α* will be called the domain of *the supposition α*. Thus in our example, the domain of the supposition having the form "*q*" is a class of which the elements have the following forms. "*q*", "*1·SNp*", "*1·Nq*", and "*p*". Besides domains of suppositions, we shall give the name "domain" to the class of all theses belonging to the system. This will be called the absolute domain [5]).

The meaning of the word "domain" like that of "system" depends upon the set of theses which are written down up to a given moment. The domains grow wider in conformity with the development of the system, for they obtain new elements. The absolute domain is assumed as existing, though empty, before the first thesis has been written.

In reality, the rules we shall give will enable us to subjoin new theses to the system. Nevertheless, for the sake of brevity, we shall use other words for expressing this fact in the formulation of the rules. So we shall say: "It is allowed to subjoin every expression satisfying some condition Φ to a given domain *D*" and by so saying we shall mean what can be expressed more exactly as follows: "Given a domain *D*, it is allowed

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5) The class la taken here in the meaning employed by A. N. W h i t e h e a d and B. R u s s e l l in Principia Mathematica, Vol. I Cambridge 1925. It is possible to understand the domain as a class of expressions in conformity with Mr. L e ś n i e w s k i' s view of a class as a material object, but in that case the subsequent explanations would have to be modified and the formulation of the rules would become more complicated.

to subjoin to the system a new thesis which contains an expression satisfying the condition Φ and which belongs to a new domain composed of the elements of the forrner domain *D* and of this expression".

This abbreviation will be applied in the following rule I.

Rule I. *To every domain D, it is allowed to subjoin an expression composed successively:*

(1) of a number not equiform with the initial number of any other element of the domain D,
(2) of a dot,
(3) of a symbol "S" and
(4) of a proposition.

In virtue of this rule, we should be able to obtain some expressions among the above written, for instance those having the forms: "*1·Sp*", *"1·1·SCpq", "2·1·Sq", "2·1·1·SNp".*

Given two different domains *D* and *D'*, where *D* is the domain of a supposition *α* and *D'* either the absolute domain or the domain of a supposition *ß* whose prefix is equiform with an initial part of the prefix of *α*, we shall say that *D* is a *subdomain* of *D'.* Thus the domain of the above mentioned supposition "*Np*" with the prefix *"2·1·1·" is* a subdomain of the domain of the supposition *"q"* with the prefix *"2·1·".* A proposition belonging to a domain will be called *valid* in every subdomain of that domain. Thus the proposition *"CpCCpqq"* is valid in the domains of suppositions "*p*", "*Cpq*", *"CNpNq", "q"* and "*Np*".

Rule II. *If in the domain D of a supposition α a proposition p is valid, it is allowed to subjoin a proposition of the form* "*Cαß*'*, to the domain whereof D is an immediate subdomain.*

Here we regard a given subdomain *D* of a domain *D'* as an *immediate subdomain* then and only then, when *D* is not the subdomain of any subdomain of *D'.* In our given example, the rule II would allow us to obtain expressions which have the forms *"1·CCpqq"* (in reference to "*1·1·SCpq"* and "*1·1·q")* and *"CCNpNqCqp"* (from "*2·SCNpNq"* and *"2·Cqp").*

The next rule is a generalisation of the usual rule of inference *(modus ponens):*

Rule III. *Given a domain D in which two propositions are valid, one of them being* α *and the other being composed successively:*

(1) of a symbol "C",

(2) of a proposition equiform with α,

(3) of a proposition β,

it is allowed to subjoin to the domain D a proposition equiform with β.

As an example of the application of this rule, we may cite the deduction of "*1·1·q*" from "*1·Sp*" and "*1·1·SCpq*".

It remains to formulate a rule embodying the principle of *reductio ad absurdum.* We can distinguish between two forms of this principle. The first form states: "Because the supposition α has led us to β and not-β, not-α must be the case". This form perhaps is more natural than the other, but it has less deductive power, as we shall see in § 3. Therefore we must use the other form of the principle, mainly: "Because the supposition not-α has led us to β and not-β, α must be the case". This form is embodied in the following structural rule:

Rule IV. *Given a domain D of a supposition composed successively of a symbol "N" and of a proposition* α, *if two propositions* β *and* γ *are valid in D such that* γ *is composed successively of a symbol "N" and of a proposition equiform with* β, *it is allowed to subjoin a proposition equiform with* α *to that domain whereof D is an immediate subdomain.*

Example: the conclusion *"2·1·*p" from the premises *"2·1·SNp, "2·1·Sq*" and *"2·1·1·Nq".*

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Thus we have formulated all the rules of our system which has the pecularity of requiring no axioms. Now we shall give some theses of this system. In order to facilitate reference to various theses, we shall classify them independently of the numbers constituting parts of theses.

Furthermore, since other systems will be developed in subsequent chapters, we shall prefix to the numbers of theses the letters "**td**" as abbreviation for αtheory of deduction". To the right of each thesis, we shall write the number of the rule used in obtaining that thesis and the numbers of theses to which we appeal. None of these numbers are parts of the theses in contrast to the numbers belonging to the prefixes.

We begin by repeating the theses which were obtained above in an intuitive way.

td	1	*1·Sp*	I	
td	2	*1·1·SCpq*	I	
td	3	*1·1·q*	III	2,1
td	4	*1·CCpqq*	II	2,3
td	5	*CpCCpqq*	II	1,4
td	6	*2·SCNpNq*	I	
td	7	*2·1·Sq*	I	
td	8	*2·1·1·SNp*	I	
td	9	*2·1·1·Nq*	III	6, 8
td	10	*2·1·p*	IV	8, 7 ,9
td	11	*2·Cqp*	II	7, 10
td	12	*CCNpNqCqp*	II	6, 11
td	13	*1·2·Sq*	I	
td	14	*1·Cqp*	II	13, 1
td	15	*CpCqp*	II	1, 14
td	16	*1·3·SNp*	I	
td	17	*1·3·1·SNq*	I	
td	18	*1·3·q*	IV	17, 1, 16
td	19	*1·CNpq*	II	16,18,
td	20	*CpCNpq*	II	1,19
td	21	*3·SCpq*	I	
td	22	*3·1·SCqr*	I	
td	23	*3·1·1·Sp*	I	
td	24	*3·1·1·q*	III	21,23
td	25	*3·1·1·r*	III	22,24
td	26	*3·1·Cpr*	II	23,25
td	27	*3·CCqrCpr*	II	22,26
td	28	*CCpqCCqrCpr*	II	21,27
td	29	*4·SCpCqr*	I	
td	30	*4·1·SCpq*	I	
td	31	*4·1·1·Sp*	I	

td	32	*4·1·1·Cqr*	III	29, 31
td	33	*4·1·1·q*	III	30, 31
td	34	*4·1·1·r*	III	32, 33
td	35	*4·1·Cpr*	II	31, 34
td	36	*4·CCpqCpr*	II	30, 35
td	37	*CCpCqrCCpqCpr*	II	29, 36
td	38	*5·SCNpp*	I	
td	39	*5·1·SNp*	I	
td	40	*5·1·*p	III	38,39
td	41	*5·p*	IV	39,40,39
td	42	*CCNppp*	II	38,41
td	43	*6·SCCpqp*	I	
td	44	*6·1·SNp*	I	
td	45	*6·1·1·Sp*	I	
td	46	*6·1·1·CNpq*	III	20,45
td	47	*6·1·1·q*	III	46,44
td	48	*6·1·Cpq*	II	45,47
td	49	*6·1·p*	III	43,48
td	50	*6·p*	IV	44,49,44
td	51	*CCCpqpp*	II	43,50

Although the last thesis does not contain any negation, we made use of the rule of *reductio ad absurdum* in proving it. It can be shown, in virtue of theorem 4 of § 3, that it is impossible to avoid that rule in this proof.

td	52	*7·St*	I	
td	53	*7·1·SNu*	I	
td	54	*7·1·1·Sp*	I	
td	55	*7·1·Cpt*	II	54, 52
td	56	*7·1·2·Su*	I	
td	57	*7·1·2·1·SNp*	I	
td	58	*7·1·2·p*	IV	57, 56, 53
td	59	*7·1·Cup*	II	56, 58

The domain having the prefix "*7·1·*" can be used for showing an interesting property of systems dealing with suppositions. In the suppositions valid in that domain, two variables appear: '*t*' and "*u*". We can assign to each of them the meaning of a constant, namely to '*t*' that of the true proposition and to "*u*" that of the false one. A precise analyzis would show that in the domain with the prefix "*7·1·*" we can obtain all those propositions and only those which we are able to deduce in the usual theory of deduction in which '*t*' and "*u*" would be constants and the suppositions belonging to the theses **td** 52 and **td** 53 would be taken as supplementary axioms. Thus the domain in question can be considered as a system of the theory of deduction with those two constants. The prefix "*7·1·*" is analogous to the assertion sign of that system.

The above example shows the analogy between domains and deductive systems. Every domain can be considered as a system having its own axioms and constants, though not every domain gives a complete system, much less an interesting one. Thus the system we were occupied with in the present chapter can be considered as composed of many systems. For this reason this system will be called the *composite system* of the theory of deduction and, in contrast to it, the usual system will be called the *simple* one.

§ 2. **The relation between the composite and the simple system of the theory of deduction**

Among the theses obtained in § 1 we find those which, being taken as a set of axioms, give the complete simple system of the theory of deduction. For instance the theses **td** 28, **td** 42 and **td** 20 are equiform with the axioms of Ł u ka s i e w i c z [6]).

The simple system of the theory of deduction having "*C*" and "*N*" as constant terms will be symbolized with capital letters **TD**. The axioms of Ł u ka s i e w i c z are taken as the first theses:

TD 1 CCpqCCqrCpr

TD 2 CCNppp

TD 3 CpCNpq

For purposes of brevity the theses of the composite system will be called theses **td**, those of the simple system theses **TD**.

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[6]) Ł u ka s i e w i c z op. cit. Elementy logiki matematycnej p. 45.

Now let Φ denote the property exhibited by those theses **TD** for which an equiform thesis **td** can be obtained. As we have seen above the axioms **TD** 1, **TD** 2 and **TD** 3 have the property Φ. Hence for proving that all theses **TD** have the same property, it is sufficient to show that Φ is a hereditary property with respect to both rules of the simple system. Suppose that α is thesis **TD** having the property Φ and $\alpha(p_1/\beta_1, p_2/\beta_2,...p_k/\beta_k)$ is the result of replacing the variables by propositions β_1, β_2, ...β_k .We can see that a thesis equiform to $\alpha(p_1/\beta_1, p_2/\beta_2, ...p_k/\beta_k)$ can be obtained in the composite system as the result of a proof analogous to that of the thesis **td** equiform to α. Only the following modifications have to be made: (1) instead of the variables p_1, p_2, ... p_k, we must always write the corresponding propositions β_1, β_2, β_k and (2) some numbers in the prefixes must be changed. Thus $\alpha(p_1/\beta_1, p_2/\beta_2, ...p_k/\beta_k)$ has the property Φ.

If now two theses **TD** having the forms "C$\alpha\beta$" and "α" have the property Φ, the thesis **TD** of the form "β" obtained from them in virtue of the rule of inference has also the property Φ: for, an equiform thesis **td** can be obtained in accordance with the rule III. Thus we see that no rule of the simple system can give us the first thesis **TD** which has not the property Φ. Hence follows the following theorem 1:

Theorem 1. *Given any thesis* **TD**, *we can obtain a thesis* **td** *equiform with the former.*

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It is obvious that such a theorem cannot be inverted, for some theses **td**, not being "propositions", are meaningless in the simple system. Their intuitive meaning can, however, be interpreted by means of a proposition which will be called the development of the given thesis **td**. Suppose that we have the following theses containing suppositions:

$$n_1{\cdot}S\ \alpha_1$$
$$n_1{\cdot}\ n_2{\cdot}S\ \alpha_2$$
$$.\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .$$
$$n_1{\cdot}\ n_2{\cdot}\ldots n_k{\cdot}S\ \alpha_k$$

where k ≥ 1. Let β be an element of the domain of the last supposition α_k. It can happen that β is identical with α_k or not.

In the first case the thesis containing β is written above, in the second case this thesis has the form

$$n_1 \cdot n_2 \cdot \ldots\, n_k \cdot \beta$$

In both cases any expression of the form

$$C\alpha_1\, C\alpha_2 \ldots\, C\alpha_k\, \beta$$

will be called the *development* of the thesis containing β. Thus we have defined the development of any thesis possessing a prefix. In the case of a thesis which is a proposition β having no prefix, the development proceeds by considering every proposition equiform with that thesis. In that case, the development can be represented by the above given general schema by putting k = 0. Now we are able to explain what it means for a simple system and a composite one to be two formal systems of the same theory. It occurs then and only then, when (1) for every thesis of the simple system an equiform thesis can be obtained in the composite one and (2) for every thesis of the composite system we can obtain a development which is a thesis of the simple system. Two such systems will be called correspondent to each other. The two systems of theory of deduction **TD** and **td** are correspondent systems, as is shown by theorem 1 and the following theorem 2.

Theorem 2. *Given* any *thesis* **td**, *we can obtain a thesis* **TD** *which is its development.*

It is known that, for any expression satisfying the method of verification through substitution of the values 1 and 0 for variables, we can obtain an equiform thesis **TD**. Hence for proving the theorem 2, it is sufficient to show that all developments of theses **td** satisfy this method of verification. Consider an arbitrary substitution, for variables, of the values: 1 standing for the truth, 0 standing for the falsehood. The development "$C\alpha_1\, C\alpha_2 \ldots\, C\alpha_k\, \beta$" receives the value 1, if at least one of the propositions $\alpha_1, \alpha_2, \ldots\, \alpha_k$ has the value 0 or if β has the value 1. Now it can be easily shown that the property of having 1 as value of the developments is hereditary with respect to all the rules of the composite system, whence it follows that all theses **td** have this property.

3. **Regarding some incomplete systems of the theory of deduction.**

The first question which we shall answer in the present chapter is: which is the simple system correspondent to that composite one in which no other rule than I, II, III holds and in which "*C*" is the only constant term of the propositions? We shall show that this simple system is the incomplete system known as the "positive logic", based on H i l b e r t' s [7]) four axioms containing no negation. Here another set of axioms will be taken, namely the following two taken from among those of F r e g e [8]), since the equivalence between the two sets of axioms has been proved by Mr. Ł u k a s i e w i c z.

PTD	1	*CpCqp*
PTD	2	*CCpCqrCCpqCpr*

In the composite system with which we shall now deal, the rules I, II and III are valid, but in rule I, we must give another meaning to the term "proposition", for propositions cannot contain the letter "*N*". This system will be symbolized by "**ptd**" as an abbreviation for the "positive theory of deduction".

ptd	1	*1·Sp*	I	
ptd	2	*1·1·Sq*	I	
ptd	3	*1: Cqp*	II	2,1
ptd	4	*CpCqp*	II	1,3
ptd	5	*2·SCPCqr*	I	
ptd	6	*2·1·SCpq*	I	
ptd	7	*2·1·1·Sp*	I	
ptd	8	*2·1·1·Cqr*	III	5,7
ptd	9	*2·1·1·q*	III	6,7
ptd	10	*2·1·1·r*	III	8,9
ptd	11	*2·1·Cpr*	III	7,10
ptd	12	*2·CCpqCpr*	II	6,11
ptd	13	*CCPCqrCCPqCpr*	II	5,12

Having obtained the theses **ptd** 4 and **ptd** 13, equiform to the axioms **PTD** 1 and **PTD** 2, *we can* prove the following

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[7]) D. H i l b e r t. *Die logischen Grundlagen der Mathematik.* Math. Annalen, Vol, 88. 1922, p. 153.

[8]) G. F r e g e. *Begriffsschrift.* Halle 1879, p. 26.

theorem by means of a demonstration analogous to that of the theorem 1.

Theorem 3. *Given any thesis* **PTD**, *we can obtain an equiform thesis* **ptd**.

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We shall be able to state that the systems **PTD** and **ptd** are correspondent, if we prove the following theorem.

Theorem 4. *Given any thesis* **ptd**, *it is possible to obtain a thesis* **PTD** *which is its development.*

The proof requires reference to some theses **PTD** which will be obtained for that purpose below. As in Mr. Ł u k a s i e w i c z' s works [9]), all the theses with exception of the axioms will be preceded by the proof lines which will indicate the applications of the rules. The numbers in those lines are the numbers of theses **PTD**.

2q/Cqp, r/p * C1qCqp — C1 — 3

PTD 3 Cpp

1p/CCpCqrCCpqCpr, q/s * C2 — 4

PTD 4 CsCCpCqrCCpqCpr

2 p/Cpq, q/CrCpq, r/CCrpCrq * C4 p/r, q/p, r/q, s/Cpq — C1 p/Cpq, q/r — 5

PTD 5 CCpqCCrpCrq

5 q/Cqp *C1 -6

PTD 6 CCrpCrCqp

5 p/CpCqr, q/CCpqCpr, r/s * C 2 — 7

PTD 7 CCsCpCqrCsCCpqCpr

5 p/CCspCsCqr, q/CCspCCsqCsr, r/CpCqr * C7 s/Csp, p/s —C5 q/Cqr, r/s — 8

PTD 8 CCpCqrCCspCCsqCsr

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* *

1 *p/Cpp* * C3 — 9

PTD 9 *CqCpp*

1 *p/CqCpp, q/r* * C9 — 10

PTD 10 *CrCqCpp*

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[9]) J. Ł u k a s i e w i c z **opera citata.**

By repeating the above process, we can obtain any thesis having the form:

I_k $Cp_k Cp_{k-1} \ldots Cp_1 p_1$ $(k \geqslant 1)$

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* *

6 r/p, p/Cqp, q/r $*$ C 1 — 11

PTD 11 CpCrCqp

6 r/p, p/CrCqp, q/s $*$ C11 — 12

PTD 12 CpCsCrCqp

Proceeding in an analogous way, we can prove any proposition

$II_{k,0}$ $CpCq_kCq_{k-1} \ldots Cq_1p$ $(k \geqslant 1)$

and then, by means of repeated use of the syllogism **PTD** 5, we can prove each thesis:

II $CCr_l\, Cr_{l-1} \ldots Cr_1pCr_l\, Cr_{l-1} \ldots Cr_1Cq_k\, Cq_{k-1} \ldots Cq_1p$

$(k \geqslant 1,\ l \geqslant 0)$

as shown by the examples below.

5 q/CsCrCqp, r/t $*$ C 12 — 13

PTD 13 CCtpCtCsCrCqp

5 p/Ctp, q/CtCsCrCqp, r/u $*$ C 13 — 14

PTD 14 CCuCtpCuCtCsCrCqp

*

* *

The proof of the third scheme containing the needed theses can be illustrated as below:

8 p/CrCpq, q/Crp, r/Crq $*$ C2 p/r, q/p, r/q — 15

PTD 15 CCsCrCpqCCsCrpCsCrq

8 p/CsCrCpq, q/CsCrp, r/CsCrq, s/t $*$ C 15 — 16

PTD 16 CCtCsCrCpqCCtCsCrpCtCsCrq

The third scheme has the form:

III_k $CCr_kCr_{k-1} \ldots Cr_1CpqCCr_kC_{k-1} \ldots Cr_1pCr_kCr_{k-1} \ldots Cr_1q$

$(k \geqslant 1)$

If *k 1,* this scheme represents **PTD** 2 with different variables only.

*

* *

Consider now the following property ψ: a thesis **ptd** is said to have the property ψ, if and only if we can obtain a thesis **PTD** which is its developement. Suppose that theses **ptd** obtained up to a given moment have the property ψ and that ζ is the next thesis **ptd** subjoined to the system. In all such cases ζ has the property ψ, as we show below.

1st case. Let ζ be a thesis obtained in virtue of the rule 1. Its development then has the form

$$C\alpha_1\, C\alpha_2 \ldots\, C\alpha_k\, \alpha_k$$

and can be obtained in the simple system by means of a substitution in the thesis I_k.

2nd case. Let ζ be a thesis which has been subjoined in virtue of the rule II. The application of that rule is possible only when some proposition P is valid in the domain of a supposition α . The development of the thesis ζ must have the form

$$(^*) \qquad C\alpha_1\, C\alpha_2 \ldots\, C\alpha_m\, \beta$$

where $\alpha_1, \alpha_2, \ldots \alpha_m$ stand for suppositions valid in the domain of α_m. The proposition β, being valid in the domain of α, must be an element (a) of the absolute domain or (b) of the domain of the supposition α_m or (c) of the domain of one of the suppositions $\alpha_1, \alpha_2, \ldots \alpha_{m-1}$. In the case (b) this development will be identical with $(^*)$, in the case (a) it will have the form of β and in the case (c) the form

$$C\alpha_1\, C\alpha_2 \ldots\, C\alpha_l\, \beta$$

where l satisfies: $1 \leqslant l < m$. In any case $(^*)$ follows from that development: in accordance with the thesis $\mathrm{II}_{m,0}$ in the case (a) and in accordance with the thesis $\mathrm{II}_{m-l,l}$ in the case (c).

3rd case. Let ζ be a thesis **ptd** obtained in virtue of the rule III. If its propositional part is β, its development must be

$$\beta$$

or

$$(^{**}) \qquad C\alpha_1\, C\alpha_2 \ldots\, C\alpha_k\, \beta$$

If β is the development of ζ, two premisses having the form of "$C\gamma\beta$" "γ"must be valid in the absolute domain and, being their own developments, they must be equiform with some theses **PTD** from which we can obtain a thesis **PTD** which is equiform with β and therefore a development of ζ. If now $(^{**})$ is the development of ζ, β is an element of the domain of the supposition α_k, and the premisses "$C\gamma\beta$" and "γ" are valid in that domain. Thus by means of the rule II we are able to obtain theses having as developments the following propositions:

$$C\alpha_1\, C\alpha_2 \ldots\, C\alpha_k\;\; C\gamma\beta$$

$$C\alpha_1\, C\alpha_2 \ldots\, C\alpha_k\;\; \gamma$$

As has been shown above it is possible to obtain such developments as theses **PTD**. Hence (**) follows by help of the thesis III_k .

Thus we see that ζ has in all cases the property ψ and we can never obtain the *first* thesis **ptd** not having the property ψ. Hence follows that all the theses **ptd** have the property ψ, and the theorem 4 is proved.

*

* *

It remains to analyze the first of the two forms of the *reductio ad absurdum* which were mentioned in § 1. We shall consider a system in which two terms "*C*" and "*N*", are constants. Its rules are I, II, III and a rule IVa formulated below.

Rule IVa. *When in the domain D of a supposition α two propositions β and γ are valid and γ has the form "$N\beta$", it is allowed to subjoin a proposition of the form "$N\alpha$" to that domain in respect to which the domain D is an immediate subdomain.*

This system (it may be referred to as "**itd**") corresponds to the simple system constructed by K o l m o g o r o f f [10]) for the purpose of embodying the laws of the intutionist logic of B r o u w e r. As axioms we shall take: the two axioms of the positive logic which are equivalent to the four axioms of H i l b e r t employed by K o l m o g o r o f f

ITD 1 CpCqp

ITD 2 CCpCqrCCpqCpr

and the axiom subjoined by K o l m o g o r o f f to those of the positive logic

[10]) A. N. K o l m o g o r o f f. 0 principie tertium non datur. Matematičeski Sbornik. Vol. XXXII, p. 651.

ITD 3 CCpqCCpNqNp

The system **ITD** is not only incomplete but does not even contain some theses belonging to the system built by H e y t i n g for the same purpose [11]). One of H e y t i n g ' s axioms,

CNpCpq

cannot be obtained as a thesis **ITD**, as can be shown by the Ł u k a s i e w i c z - B e r n a y s method with help of the following matrix:

C	0	1	N
0	1	1	1
1	0	1	1

Theorem 5. *Given any thesis* **ITD**, *it is possible to obtain a thesis* **itd** *equiform with it.*

Such a theorem can be proved as was done für the theorems 1 and 3. Theses **itd** equiform to the axioms **ITD** 1 and **ITD** 2 can be obtained on the same way as in the system **ptd**. The proof of a thesis equiform to **ITD** 3 is given below.

itd	1	1·SCpq	I	
itd	2	1·1·SCpNq	I	
itd	3	1·1·1·SP	I	
itd	4	1·1·1·q	III	1,3
itd	5	1·1·1·Nq	III	2,3
itd	6	1·1·Np	IVa	3,4,5
itd	7	1·CCpNqNp	II	2,6
itd	8	CCpqCCpNqNp	II	1,7

Theorem *6. Given any thesis* **itd**, *it is possible to obtain a thesis* **ITD** *which is its development.*

The demonstration is analogous to that of theorem 4. All theses **PTD** used in it have their equiform theses **ITD**, because all axioms of the system **PTD** are axioms **ITD**. Thus the schemes of theses I_k, $\mathrm{II}_{k,l}$, and III_k, can be applied in order to show that none of the rules I, II, III can lead to the *first* among theses **itd** which cannot have any thesis **ITD** as deve-

[11]) A. H e y t i n g. *Die formalen Regeln der intuitionistischen Logik.* Sitzungsber. d. Preuss. Ak. d. Wiss. 1930. Phys.-Math. Kl. p. 45 ss.

lopment. It remains to prove that the same property holds for the rule IVa. Let

$$C\alpha_1\, C\alpha_2 \ldots\, C\alpha_k\alpha_k$$

be the development of the thesis containing the supposition α_k which has led us to the contradiction "β" and "$N\beta$". Since those propositions are valid in the domain of a_k we can prove, as was done in the demonstration of the theorem 4, that the following theses **ITD** can be obtained

(***) $$C\alpha_1\, C\alpha_2 \ldots\, C\alpha_k\beta$$

and

(****) $$C\alpha_1\, C\alpha_2 \ldots\, C\alpha_k N\beta$$

If k = 1, the thesis just now subjoined in virtue of the rule IVa has the form

$$C\alpha_1\, C\alpha_2 \ldots\, C\alpha_{k-1} N\alpha_k$$

and its development which has the same form, can be obtained as a thesis **ITD**, by means of **ITD** 3. If k > 1, the development of the subjoined thesis can be obtained from (***) and (****) by means of the thesis

IV_k $$CCp_k\, Cp_{k-1} \ldots Cp_1 q CCP_k\, Cp_{k-1} \ldots Cp_1 Nq Cp_k Cp_{k-1} \ldots Cp_2 Np_1$$

This scheme becomes **ITD** 3, when k = 1. As to the theses represented by this scheme for other values of k, they can be obtained from **ITD** 3 by use of the thesis **PTD 8.**

§ 4. The extended theory of deduction.

The following question arises: By what means is it possible to transform the more complicated theories into systems in which our rules may hold? In those theories, beyond the rules which replace the theory of deduction, some others are required; they are those concerning the apparent variable. As to them, in the present and the next chapters we attempt to transform them into structural rules adapted to the symbolism of composite systems. For the present, we shall take as a basis the extended theory of deduction [12]). Besides all the symbols met with till now, that theory contains one mo-

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[12]) J. Ł u k a s i e w i c z op. cit. *Elementy* § 8 - Ł u k a s i e w i c z and A. T a r s k i op. cit. § 5.

re — the general quantifier "Π"' which appears in the connexions "$\Pi p\alpha$", "$\Pi q\alpha$" *etc.,* α being a proposition which in this case may or not contain a variable of the form "*p*", "*q*" *etc.* ""$\Pi p\alpha$" may be read: "α, whatever a proposition p may be" or in short: "for every p, α (holds)", because. in the system in question, we deal with no other variables than propositional ones. We shall present examples of reasoning with the use of "Π", leaving the symbolism of the composite systems unchanged.

etd	1	$1{\cdot}S\Pi qCqp$		

The above supposition means: "*Cqp*, whatever proposition *q* may be". Hence follows "*CCppp*", because "*Cpp*" is a proposition. Thus we may write

etd	2	$1{\cdot}$CCppp		

In fact, the rule of substitution which we shall formulate will enable us to accept the above as a thesis. In accordance to the rules of suppositions, we can deduce

etd	3	$2{\cdot}Sp$	I	
etd	4	Cpp	II	3,3
etd	5	$1{\cdot}p$	III	2,4
etd	6	$C\Pi qCqpp$	II	1,5
etd	7	$2{\cdot}1{\cdot}Sq$	I	
etd	8	$2{\cdot}Cqp$	II	7,3

"*Cqp*" is valid in a domain in which no supposition concerning *"q"* is valid. In that thesis, "*q*" is a quite arbitrary proposition and all the results we have obtained concerning it could be accepted for any other proposition. That is the intuitive reason for which the rule VI will allow us to write

etd	9	$2{\cdot}\Pi qCqp$		

Hence

etd	10	$Cp\Pi qCqp$	II	3,9

* *

The rule of substitution in the system **etd** will be formula-as follows.

Rule V. *To any domain D in which a proposition α is valid, where α is composed successively:*

(1) of a general quantifier "Π*"*

(2) of a variable ζ and

(3) of a proposition ß,

it is allowed to subjoin a proposition γ obtained from ß by means of substitution for variables bound to ζ .

The above condition concerning *γ* must be explained by the following detailed description:

As to their form, ß and γ differ only in this, that instead of all those variables of ß which are bound to ζ, γ contains propositions

(1) equiform with one another and

(2) having the following property: all real variables of those propositions are real variables of γ.

We must still explain some expressions in the above by means of structural description. A variable *ζ* is said to be a *real variable* of a proposition *α* then and only then, when it does not belong to any significant part [13]) of *α* beginning with a quantifier *"Π"* und a variable equiform with *ζ*.

A variable *η* *is bound* to a variable *ζ* then und only then, when they are equiform with each other and they belong to a proposition composed successively:.

(1) of a quantifier "Π",

(2) of the variable *ζ,*

(3) of a proposition in which *η* is a real variable.

Rule VI. *lf a proposition α is valid in the domain D, it is allowed to subjoin a proposition of the form " Πζα" to the domain D, provided that ζ is a variable not equiform with any real variable of a supposition valid in D.*

The rules I, II, III, IV remain in force, but as to I the notion of the "proposition" is altered. With help of all our rules we can obtain the following theses

etd	11	**2**·**2**·SCpq	I	
etd	12	**2**·**2**·q	III	11,3
etd	13	**2**·CCpqq	II	11,12
etd	14	CpCCpqq	II	3,13
etd	15	ΠqCpCCpqq	VI	14

[13]) The "part" is understood here in such a way that it can be identical with the whole.

etd	16	ΠpΠqCpCCpqq	VI	15
etd	17	ΠpCpp	VI	4

*

*   *

Analogously we can obtain all theses of the usual theory of deduction where each is preceded by quantifiers. We shall now present a process which will illustrate how it is possible to apply definitions in the composite system.

etd	18	***3***·*SCu*Π*pp*	I	
etd	19	***3·1***·*SC*Π*ppu*	I	
etd	20	***3·1·1***·*SCtNu*	I	
etd	21	***3·1·1·1***·*SCNut*	I	

The above four suppositions can be considered as the definitions of two constant terms: "*u*" being equivalent to "Π*pp*" and "*t*" equivalent to "*Nu*". The domain with the prefix "***3·1·1·1***" gives us the enlarged theory of deduction with those constants.

etd	22	**3**·1·1·1·1·*Su*	I	
etd	23	**3**· 1·1·1·1·Π*pp*	III	18,22
etd	24	**3**·1·1·1·1·*p*	V	23
etd	25	**3**·1·1·1·*Cup*	II	22,24
etd	26	**3**·1·1·1·Π*pCup*	VI	25

The false proposition '*u*' implies anything.

etd	27	**3**·1·1·1·2·*Sp*	I	
etd	28	**3**·1·1·1·2·1·*SNt*	I	
etd	29	**3**·1·1·1·2·1·1·*SNu*	I	
etd	30	**3**·1·1·1·2·1·1·*t*	III	21,29
etd	31	**3**·1·1·1·2·1·*u*	IV	29,30,28
etd	32	**3**1·1·1·2·1·Π*pp*	III	18,31
etd	33	**3**· 1·1·1·2·1·*Np*	V	32
etd	34	**3**· 1·1·1·2·*t*	IV	28,27,33
etd	35	**3**·1·1·1·*Cp*t	II	27,34
etd	36	**3**·1·1·1·Π*pCpt*	VI	35

The true proposition '*t*' is implied by anything.

etd	37	**3**·1·1·*CCNut*Π*pCpt*	II	21,36
etd	38	**3**·1·*CCtNuCCNut*Π*pCpt*	II	20,37
etd	39	**3**·1·Π*tCCtNuCCNut*Π*pCpt*	VI	38

etd	40	**3**·1·*CCNuNuCCNuNu*Π*pCpNu*	V	39
etd	41	**3**·1·CNuNu	V	17
etd	42	**3**·1·CCNuNuΠpCpNu	III	40,41
etd	43	**3**·1·ΠpCpNu	III	42,41

The above process, leading us from **etd** 36 to **etd** 43 shows how it is possible to carry a proposition containing a defined term out of the domain of the definition by replacing at the same time the *definiendum "t"* by the *definiens* "*Nu*". With the same process, we can transport the proposition "Π*pCpNu*" to the absolute domain by replacing "*u*" by "Π*pp*":

etd	44	**3**·1·*CC*Π*ppu*Π*pCpNu*	II	19,43
etd	45	***CC****u*Π*ppCC*Π*ppu*Π*pCpNu*	II	18,44
etd	46	Π*uCCu*Π*ppCC*Π*ppu*Π*pCpNu*	VI	45
etd	47	*CC*Π*pp*Π*ppCC*Π*pp*Π*pp*Π*pCpN*Π*pp*	V	46
etd	48	*C*Π*pp*Π*pp*	V	17
etd	49	*CC*Π*pp*Π*pp*Π*p*C*pN*Π*pp*	III	47,48
etd	60	Π*pCpN*Π*pp*	III	49,48

The system **ctd** corresponds to the simple system of the enlarged theory of deduction. The needed proofs are analogous to those of the theorems **1** and **2.**

§ 5. Application to the calculus of functions.

We shall now consider a theory analogous to the theory of apparent variable of the *Principia Mathematica* and to H i l b e r t ' s calculus of functions. Contrary to the preceding chapter, we shall deal now with the individual apparent variable rather than with the propositional one. The individual variables will be symbolized by small Latin letters "*x*", "*y*" *etc.* and will be the arguments of functions having propositions as values. The symbols of those functions are the small Greek letters "*φ*", "*ψ*" *etc.* The arguments will follow those letters; brackets are superfluous [14]). The quantifiers "Π" ought to be followed by an individual variable and a proposition *e.g.* as in "Π*xφx*" or

[14]) Brackets are omitted in such functions by J. H e r b r a n d. See his *Recherches sur la théorie de démonstration.* Comptes Rendus d.s. de la Soc. des Lettres et des Science de Varsovie. XXIII. 1930.

"$\Pi x C p q$" which are to be read: "for every (indvidual) x, φx holds" and "for every (individual) x, *Cpq* holds".

The rules I, II, III, IV, V and VI, when adapted to such a theory, would give a system correspondent to some simple system differing from those of *Principia Mathematica* and of H i l b e r t only as a consequence of the fact that the notion of significance is different. In such a system, we should be able to have a thesis of the following form:

$$C \Pi x \varphi x N \Pi x N \varphi x$$

The intuitive meaning of "$N \Pi x N \varphi x$" *is:* "for some x, φx holds". The above thesis therefore means: "If for every x, φx, then for some x, φx". In the null field of individuals *(Individuenbereich), i.e.* under the supposition that no individual exists in the world, this proposition is false. Thus the system states the existence of at least one individual. But whether individuals exist or not, it is better to solve this problem through other theories. We shall present therefore a system of the calculus of functions, where all the theses will be satisfied in the null field of individuals.

In that system, we must avoid any thesis which is a proposition with real individual variables, for such theses lead us to assume others requiring the existence of individuals. As to the notion of real variables in the composite system, the circumstances are quite different from those of the simple system. Symbols of variables which are not apparent variables do not merit the name of variable at all. We deal with such a term as with a given constant, though it is neither a primitive term nor a defined one. It is a constant the meaning of which, although undefined, remains unaltered through the whole process of reasoning. In practice, we often introduce such undefined constants in the course of a proof. For example we say: "Consider an arbitrary *x*" and then we deduce propositions which can be said to belong to the *scope of constancy* of the symbol "*x*". This process of reasoning will be applied in our system. We shall give an example of such a reasoning in the calculus of functions.

Suppose $\Pi x \Pi y \varphi xy$. *Consider* now an arbitrary individual z. According to our supposition, "$\Pi y \varphi xy$" holds whatever individual x may be, hence $\Pi y \varphi zy$ holds too. Repeating the same process in respect to y, we obtain φzz. We have obtained this result for an arbitrary z, therefore our result must be "for every z, φzz holds" i.e. $\Pi z \varphi zz$. This is a consequence of the supposition "$\Pi x \Pi y \varphi xy$", so we can take as a thesis

$$C\Pi x \Pi y \varphi xy \Pi z \varphi zz$$

We shall repeat the same proof in symbols.

cf 1 $1 \cdot S\Pi x \Pi y \varphi xy$

Now we shall write the expression "Consider an arbitrary z"

cf 2 $1 \cdot 1 \cdot Tz$

"T" is here a new constant analogous to the symbol of supposition "S".

The arbitrary constant "z" will be called the *term* and the scope of its constancy the *domain of that term.* Further steps of our proof can be easily formalized.

cf 3 $1 \cdot 1 \cdot \Pi y \varphi zy$

cf 4 $1 \cdot 1 \cdot \varphi zz$

cf 5 $1 \cdot \Pi z \varphi zz$

cf 6 $C\Pi x \Pi y \varphi xy \Pi z \varphi zz$

We shall make use of the abbreviated modes of speaking which were introduced in the § 1 in connection with suppositions and their domains, adapting them now to the terms and their domains. Thus in any domain *D,* we shall consider as *valid* any term whose prefix is equiform to an initial part of the prefix of the thesis containing the supposition or the term belonging to *D.* The new rules will be formulated below.

Rule Va. *It is allowed, to any domain D in which*

(1) a term ζ *and*

(2) a proposition composed of a quantifier "Π", *of a variable* η *and of a proposition* α

are valid, to subjoin a proposition which differs, as to its form, from α *only in this respect, that all variables bound to* η *are replaced by symbols equiform with* ζ *no one of which is an apparent variable.*

Rule Vla. *If in the domain D of a term ζ a proposition α is valid, it is allowed to subjoin a proposition of the form "Πζα" to that domain whereof the domain D is an immediate subdomain.*

Rule VII. *Given a domain D, it is allowed to subjoin to it any expression composed successively:*

(1) of a number not equiform with the initial number of any element of the domain D,
(2) of a dot,
(3) of the symbol "T" and
(4) of a term not equiform with any term valid in the domain D.

The rules formulated in § 1 remain unaltered for the sy-stern of with the exception of rule I which must be transformed into rule Ia.

Rule Ia. *Given a domain D, it is allowed to subjoin to it any expression composed successively:*

(1) of a number not equiform with the initial number of any element of the domain D,
(2) of a do t,
(3) of the symbol "S",
(4) of a proposition significant in the domain D.

We regard a proposition α, *as significant in the domain D,* if every real variable of α is equiform with some term valid in D.

*

* *

We shall give some further examples of theses of the system cf.

cf	7	*1·1·1·Tv*	VII	
cf	8	*1·1·1·φzy*	Va	3, 7
cf	9	*1·1·* Π*vφzv*	VIa	7, 8
cf	10	*1·* Π*z*Π*vφzv*	VIa	2, 9
cf	11	*1·2·Tx*	VII	
cf	12	*1·2·1·Ty*	VII	

cf	13	*1·2·1* $\Pi v\varphi yv$	Va	10, 12
cf	14	*1·2·1* φyx	Va	13, 11
cf	15	*1·2·* $\Pi y\varphi yx$	VIa	12, 14
cf	16	*1·*$\Pi x\Pi y\varphi yx$	VIa	11, 15
cf	17	$C\Pi x\Pi y\varphi xy\Pi x\Pi y\varphi yx$	II	1, 16
cf	18	*2·*$S\Pi xC\varphi x\psi x$	Ia	
cf	19	*2·1·*$S\Pi x\varphi x$	Ia	
cf	20	*2·1 1·*Tx	VII	
cf	21	*2·1·1·*$C\varphi x\psi x$	Va	18, 20
cf	22	*2·1·1·*φx	Va	19, 20
cf	23	*2·1·1·*ψx	III	21, 22
cf	24	*2·1·*$\Pi x\psi x$	VIa	20, 23
cf	25	*2·* $C\Pi x\varphi x\Pi x\psi x$	II	19, 24
cf	26	$C\Pi xC\varphi x\psi xC\Pi x\varphi x\Pi x\psi x$	II	18, 25
cf	27	*3·*Sp	Ia	
cf	28	*3·1·*Tx	VII	
cf	29	*3·* Πxp	VIa	28, 27
cf	30	$Cp\Pi xp$	II	27, 29

The rules of the composite systems can be applied to different logical or mathematical systems. In such cases, it can happen that new rules may be required. For instance, it we want to build the composite system of the theory of deduction having besides "*C*" and "*N*" the new constant term of *conjunction* (logical product), it is sufficient to give three new rules. The first would permit us to subjoin to a domain a conjunction composed of propositions equiform with some propositions valid in that domain, and the others would allow to subjoin a proposition equiform with the first and a proposition equiform with the second member of a valid conjunction.

By building composite systems containing variables of different kinds from those already taken into consideration, we must suitably adapt the rule of substitution, and either the rule VI or the two rules Vla and VII to the new variables.

As to the application in the mathematical theories, we can expect that the composite systems of logic will be more suited to the purposes of formalizing practical proofs, than are the simple ones. The use of the theses of the theory of deduction for that purpose is so burdensome that it is avoided even by the authors of logical systems. In more complicated theories the use of theses would be completely unproductive.